

Self-Dual Model Coupled to Bosons

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Abstract

In this paper we investigated the dynamics, at the quantum level, of the self-dual field minimally coupled to bosons. In this investigation we use the Dirac bracket quantization procedure to quantize the model. Also, the relativistic invariance is tested in connection with the elastic boson-boson scattering amplitude.

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The quantum field theory in $(2 + 1)$ dimensions has been provided for explaining the quantum Hall effect and the high temperature superconductivity. Besides, interesting aspects like exotic statistics, fractionary spin and insights about existence of massive gauge field is present in the context of three dimensions models [1].

Of the gauge theories, the self-dual theories deserve special attention. Self duality refers to theories in which the interactions have particular forms and special strengths such that the equations of motion reduce from second to first order differential equations. In the context of the electrodynamics self-dual (SD) model, originally proposed by Townsend, Pilch and Van Nieuwenhuizen [2] has been the object of several investigations. In [3,4] it was demonstrated the equivalence, on semiclassical level, between free SD with Maxwell Chern-Simons (MCS). This equivalence was also observed in the level of the Green functions [5].

The exact equivalence between the self-dual model minimally coupled to a Dirac field and the MCS model with non-minimal magnetic coupling to fermions has been studied by Gomes et al [6]. Canonical quantization of the self-dual model coupled to fermions has been studied by Girotti [7]. In this study, one has observed that two new interactions terms arise, which are local in space and time and are non-renormalizable by power counting. Relativistic invariance is tested in connection with the elastic fermion-fermion scattering amplitude.

In this paper we will study the self-dual model coupled to boson where we are using the Dirac bracket quantization procedure to quantize it. Lorentz invariance is tested in connection with the elastic boson-boson scattering amplitude. In this case, we demonstrate that the combined action of the non-covariant pieces that make up the interaction in the Hamiltonian, can be replaced by the minimal covariant field-current interaction.

We adopt the following Heaviside-Lorentz units, and put $\hbar = c = 1$. The metric tensor is $g^{\mu\nu} = \text{diag}(1, -1, -1)$ and antisymmetric tensor $\epsilon^{\mu\nu\rho}$ is normalized as $\epsilon^{012} = 1$. Also, we have considered $\epsilon^{ij} = \epsilon^{0ij}$.

The Lagrangian that describe the self-dual field coupled to bosons is written as

$$\mathcal{L} = -\frac{1}{2m}\epsilon^{\mu\nu\rho}(\partial_\mu f_\nu)f_\rho + \frac{1}{2}f^\mu f_\mu + (D_\mu\phi)^*(D^\mu\phi) - M^2\phi^*\phi \quad (1)$$

where the covariant derivative is given by $D_\mu = \partial_\mu + \frac{ig}{m}f_\mu$, f_μ is the self-dual field and ϕ is the charged scalar field.

The momenta canonically conjugated are

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 f_\alpha)} = -\frac{1}{2m}\epsilon_{0\alpha\rho}f^\rho, \quad (2)$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi^* - \frac{ig}{m}f^0 \phi^*, \quad (3)$$

$$\Pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial^0 \phi + \frac{ig}{m}f^0 \phi. \quad (4)$$

The primary constraints are

$$P_0 = \pi_0 \approx 0, \quad (5)$$

$$P_i = \pi_i + \frac{1}{2m}\epsilon_{ij}f^j \approx 0, \quad (6)$$

where the sign of weak equality (\approx) is used in the sense of Dirac [8,9]. The canonical Hamiltonian density is given by

$$\mathcal{H} = \pi_i \partial^0 f^i + \Pi \partial^0 \phi + \Pi^* \partial^0 \phi^* - \mathcal{L}, \quad (7)$$

then,

$$\begin{aligned} \mathcal{H} = & \Pi^* \Pi - \partial_k \phi^* \partial^k \phi + M^2 \phi^* \phi + \frac{ig}{m}(f_k \phi^* \partial^k \phi - \partial_k \phi^* f^k \phi) \\ & - \frac{g^2}{m^2} f_k f^k \phi^* \phi - \frac{1}{2} f^\mu f_\mu + f^0 \left[\frac{1}{m} \epsilon_{ij} f^0 \partial^i f^j - \frac{ig}{m}(\Pi \phi - \Pi^* \phi^*) \right]. \end{aligned} \quad (8)$$

The primary Hamiltonian is

$$H_P = \int d^2x (\mathcal{H} + U^0 P_0 + U^i P_i). \quad (9)$$

where U^0 and U^i are the Lagrange multipliers.

Imposing the consistency conditions to the constraint

$$\dot{P}_0 = \{P_0, H_P\}_P = \{\pi_0(\vec{x}), H_P(\vec{y})\}_P \approx 0, \quad (10)$$

we find the secondary constraint

$$S = f^0 - \frac{1}{m}\epsilon_{ij}\partial^i f^j + \frac{ig}{m}(\Pi\phi - \Pi^*\phi^*) \approx 0. \quad (11)$$

Imposing again the same condition of consistency to the constraints P_i and S we can verify that no more constraint arise. So we can determine the Lagrange multipliers and all constraints are second class. Following the Dirac bracket quantization procedure we get the commutation relation in equal time of the dynamics variables

$$[f^0(\vec{x}), f^j(\vec{y})] = i\partial_x^j \delta(\vec{x} - \vec{y}), \quad (12)$$

$$[f^k(\vec{x}), f^j(\vec{y})] = -im\epsilon^{kj}\delta(\vec{x} - \vec{y}), \quad (13)$$

$$[f^0(\vec{x}), \pi_k(\vec{y})] = -\frac{i}{2m}\epsilon_{kj}\partial_x^j \delta(\vec{x} - \vec{y}), \quad (14)$$

$$[f^j(\vec{x}), \pi_k(\vec{y})] = \frac{i}{2}g_k^j \delta(\vec{x} - \vec{y}), \quad (15)$$

$$[\pi_j(\vec{x}), \pi_k(\vec{y})] = -\frac{i}{4m}\epsilon_{jk}\delta(\vec{x} - \vec{y}), \quad (16)$$

$$[f^0(\vec{x}), \phi(\vec{y})] = -\frac{g}{m}\phi(\vec{x})\delta(\vec{x} - \vec{y}), \quad (17)$$

$$[f^0(\vec{x}), \phi^\dagger(\vec{y})] = \frac{g}{m}\phi^\dagger(\vec{x})\delta(\vec{x} - \vec{y}), \quad (18)$$

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta(\vec{x} - \vec{y}), \quad (19)$$

$$[\phi^\dagger(\vec{x}), \Pi^\dagger(\vec{y})] = i\delta(\vec{x} - \vec{y}). \quad (20)$$

and all other commutators are vanish.

The Hamiltonian that describe the quantum dynamics of the system is written as

$$\begin{aligned}
H = \int d^2x & \left[\Pi^\dagger \Pi - \partial_k \phi^\dagger \partial^k \phi + M^2 \phi^\dagger \phi \right. \\
& \left. + \frac{ig}{m} (f^k \phi^\dagger \partial_k \phi - \partial_k \phi^\dagger f^k \phi) - \frac{g^2}{m^2} f_k f^k \phi^\dagger \phi + \frac{1}{2} f^0 f^0 + \frac{1}{2} f^i f^i \right].
\end{aligned} \tag{21}$$

where we have considered the Wick order of the operators. We can simplify the Hamiltonian eliminating the operator f^0 using the condition that is described by the Eq.(11) so that it takes the form

$$H^I = H_0^I + H_{int}^I, \tag{22}$$

where

$$\begin{aligned}
H_0^I = \int d^2x & \left[\frac{1}{2m^2} \epsilon^{ij} \epsilon^{kl} (\partial_i f_j^I) (\partial_k f_l^I) + \frac{1}{2} f_i^I f_i^I \right] \\
& + \int d^2x \left[\Pi^{I\dagger} \Pi^I - \partial_k \phi^{I\dagger} \partial^k \phi^I + M^2 \phi^{I\dagger} \phi^I \right],
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
H_{int}^I = \int d^2x & \left[\frac{ig}{m} (f^{Ik} \phi^{I\dagger} \partial_k \phi^I - \partial_k \phi^{I\dagger} f^{Ik} \phi^I) \right. \\
& - \frac{g^2}{m^2} f_k^I f^{Ik} \phi^{I\dagger} \phi^I - \frac{ig}{m^2} \epsilon^{kl} \partial_k f_l^I (\Pi^I \phi^I - \Pi^{I\dagger} \phi^{I\dagger}) \\
& \left. - \frac{g^2}{2m^2} (\Pi^I \phi^I - \Pi^{I\dagger} \phi^{I\dagger}) (\Pi^I \phi^I - \Pi^{I\dagger} \phi^{I\dagger}) \right].
\end{aligned} \tag{24}$$

The superscript I denotes field operators belonging to the interaction picture.

The rules of commutations relations in equal times obeyed by operators of field in interaction pictures are exactly the equations (12)-(20). The motion equations that satisfy the operators ϕ^I e $\phi^{I\dagger}$ are,

$$\partial_0 \phi^I = i[H_0^I, \phi^I] = \Pi^\dagger, \tag{25}$$

$$\partial_0 \phi^{I\dagger} = i[H_0^I, \phi^{I\dagger}] = \Pi. \tag{26}$$

and the correspondent propagator of bosons $\Delta(p)$ in the momentum space is

$$\Delta(p) = \frac{i}{p^2 - M^2 + i\epsilon}. \tag{27}$$

The Feynman propagator of the self-dual field $f_i^I, i = 1, 2$ is given by

$$D_{lj}(k) = \frac{i}{k^2 - m^2 + i\epsilon}(-m^2 g_{lj} + k_l k_j - im\epsilon_{lj} k_0) = D_{jl}(-k) \quad (28)$$

as has been obtained by Girotti [7].

Finally, the Hamiltonian of interactions described in terms of the fundamental fields is written as

$$\begin{aligned} H_{int}^I = \int d^2x & \left[\frac{ig}{m}(f^{Ik}\phi^{I\dagger}\partial_k\phi^I - \partial_k\phi^{I\dagger}f^{Ik}\phi^I) \right. \\ & - \frac{g^2}{m^2}f_k^I f^{Ik}\phi^{I\dagger}\phi^I - \frac{ig}{m^2}\epsilon^{kl}\partial_k f_l^I(\partial_0\phi^{I\dagger}\phi^I - \phi^{I\dagger}\partial_0\phi^I) \\ & \left. - \frac{g^2}{2m^2}(\partial_0\phi^{I\dagger}\phi^I - \phi^{I\dagger}\partial_0\phi^I)(\partial_0\phi^{I\dagger}\phi^I - \phi^{I\dagger}\partial_0\phi^I) \right]. \end{aligned} \quad (29)$$

Observe that the equation (29) contain four terms. The first term is the spatial part of the field-current interaction. The third term is the magnetic field interacting with temporal component of the current. Whereas the second term is the spatial part of the gauge-boson field interaction and the fourth term is the interaction of a kind of temporal components of currents. Unlike the case of MCS minimally coupled to bosons, these extra terms are strictly local in space-time. Also, they are non-renormalizable by power counting.

The next step we will verify the Lorentz invariance of the theory. To do this, we will evaluate the contribution of order g^2 to the lowest order elastic boson-boson scattering amplitude which can be grouped into four different kind of terms,

$$S^{(2)} = \sum_{\alpha=1}^4 S_{\alpha}^{(2)}, \quad (30)$$

where

$$\begin{aligned} S_1^{(2)} = \frac{g^2}{2m^2} \int \int d^3x d^3y & \langle \varphi_f | T \{ : \{ \phi^\dagger(x) \partial_j \phi(x) - \partial_j \phi^\dagger(x) \phi(x) \} f^j(x) : \\ & \times : \{ \phi^\dagger(y) \partial_l \phi(y) - \partial_l \phi^\dagger(y) \phi(y) \} f^l(y) : \} | \varphi_i \rangle, \end{aligned} \quad (31)$$

$$\begin{aligned} S_2^{(2)} = -\frac{g^2}{m^3} \int \int d^3x d^3y & \langle \varphi_f | T \{ : \{ \phi^\dagger(x) \partial_j \phi(x) - \partial_j \phi^\dagger(x) \phi(x) \} f^j(x) : \\ & \times : \{ \epsilon_{il} \partial^i f^l(y) (\partial_0 \phi^\dagger(y) \phi(y) - \phi^\dagger(y) \partial_0 \phi(y)) \} : \} | \varphi_i \rangle, \end{aligned} \quad (32)$$

$$S_3^{(2)} = \frac{g^2}{2m^4} \int \int d^3x d^3y \langle \varphi_f | T \{ : \{ \epsilon_{kj} \partial^k f^j(x) (\partial_0 \phi^\dagger(x) \phi(x) - \phi^\dagger(x) \partial_0 \phi(x)) \} :$$

$$\times : \{ \epsilon_{ii} \partial^i f^l(y) (\partial_0 \phi^\dagger(y) \phi(y) - \phi^\dagger(y) \partial_0 \phi(y)) \} : | \varphi_i \rangle, \quad (33)$$

$$S_4^{(2)} = \frac{ig^2}{2m^2} \int \int d^3x d^3y \delta(x-y) < \varphi_f | T \{ : \{ \partial_0 \phi^\dagger(x) \phi(x) - \phi^\dagger(x) \partial_0 \phi(x) \} : \\ \times : \{ \partial_0 \phi^\dagger(y) \phi(y) - \phi^\dagger(y) \partial_0 \phi(y) \} : \} | \varphi_i \rangle. \quad (34)$$

Here, T is the chronological ordering operator, whereas $|\varphi_i\rangle$ and $\langle\varphi_f|$ denote the initial and final state of the reaction, respectively. For the case under analysis, both $|\varphi_i\rangle$ and $\langle\varphi_f|$ are two-boson states.

In terms of the initial (p_1, p_2) and the final momenta (p'_1, p'_2) , the partial amplitude are

$$S_1^{(2)} = -g^2 N_p (2\pi)^3 \delta^3(p'_1 + p'_2 - p_1 - p_2) \{ (p'_1 + p_1)_j (p'_2 + p_2)_l \frac{1}{m^2} D^{jl}(k) + p_1 \leftrightarrow p_2 \} \quad (35)$$

$$S_2^{(2)} = -g^2 N_p (2\pi)^3 \delta^3(p'_1 + p'_2 - p_1 - p_2) \{ (p'_1 + p_1)_j (p'_2 + p_2)_0 \frac{1}{m^2} \Gamma^j(k) \\ + (p'_1 + p_1)_0 (p'_2 + p_2)_j \frac{1}{m^2} \Gamma^j(-k) + p_1 \leftrightarrow p_2 \} \quad (36)$$

$$S_3^{(2)} = -g^2 N_p (2\pi)^3 \delta^3(p'_1 + p'_2 - p_1 - p_2) \{ (p'_1 + p_1)_0 (p'_2 + p_2)_0 \frac{1}{m^2} \Lambda(k) + p_1 \leftrightarrow p_2 \} \quad (37)$$

$$S_4^{(2)} = -g^2 N_p (2\pi)^3 \delta^3(p'_1 + p'_2 - p_1 - p_2) \{ (p'_1 + p_1)_0 (p'_2 + p_2)_0 \frac{i}{m^2} + p_1 \leftrightarrow p_2 \} \quad (38)$$

where

$$\frac{1}{m^2} D^{jl}(k) = \frac{i}{k^2 - m^2 + i\epsilon} \left(-g^{jl} + \frac{k^j k^l}{m^2} - \frac{i}{m} \epsilon^{jl} k_0 \right), \\ \frac{1}{m^2} \Gamma^j(k) = \frac{i}{k^2 - m^2 + i\epsilon} \left(\frac{i}{m} \epsilon^{jl} k_l + \frac{k^j k^0}{m^2} \right), \\ \frac{1}{m^2} \Lambda(k) = \frac{i}{k^2 - m^2 + i\epsilon} \left(-\frac{k^l k_l}{m^2} \right), \quad (39)$$

$$k \equiv (p'_1 - p_1) = (p'_2 - p_2) \quad (40)$$

is the momentum transfer. Substituting the Eqs.(35)-(38) into Eq.(30) we find

$$S^{(2)} = -g^2 N_p (2\pi)^3 \delta^3(p'_1 + p'_2 - p_1 - p_2) \{ (p'_1 + p_1)_\mu (p'_2 + p_2)_\nu \frac{1}{m^2} D^{\mu\nu}(k) + p_1 \leftrightarrow p_2 \} \quad (41)$$

where

$$\frac{1}{m^2} D^{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2} + \frac{i}{m} \epsilon^{\mu\nu\alpha} k_\alpha \right), \quad (42)$$

is the propagator of the self-dual field f^μ . As can be seen the amplitude $S^{(2)}$ is the scalar of Lorentz. Observe, also, that the theory has passed in the test of the relativistic invariance. On the other hand, in the tree approximation is allowed to replace all the non-covariant terms in H_{int}^I , Eq.(29), by the minimal covariant interaction $\frac{g}{m}J_\mu^I f^\mu$. Where J_μ^I is given by

$$J_\mu^I = i(\phi^{I*}\partial_\mu\phi^I - \partial_\mu\phi^{I*}\phi^I) - \frac{g}{m}f_\mu^I\phi^{I*}\phi^I \quad (43)$$

Observe that the high energy behavior of the propagator in Eq.(42) is radically different from the MCS theory in the Landau gauge [10],

$$D_L^{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \frac{im}{k^2} \epsilon^{\mu\nu\alpha} k_\alpha \right), \quad (44)$$

and therefore, the self-dual model coupled to bosons is a non-renormalized theory as we have noted previously by power counting.

Finally, we conclude in this paper that the SD model minimally coupled to bosons bears no resemblance with the renormalizable model defined by the MCS field minimally coupled to bosons. The equivalence between SD and MCS when coupled to bosons is under investigation.

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